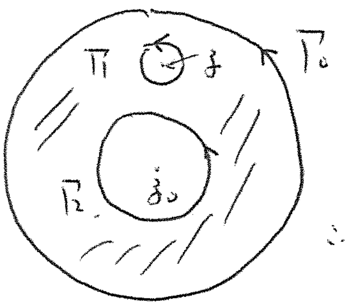
Lectures 14 and 15.

In these lectures we showed three applications of Cauchy - Gousat Thms and Cauchy integral formula.

- Liouville's Thm and its general version
- maximum moduli Thm.
- Taylor Series expansion.

By application. we apply Liouville's and maximum moduli Thm to prove the fundamental Thm of algebra. After Taylor series expansion. we proceed to show the ~~Taylor~~ Laurent series expansion on annulus.



$\frac{f(\zeta)}{\zeta - z}$ is analytic on shaded region

$$\therefore \int_{\Gamma_0} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= 2\pi i f(z) + \int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i}\int_{\Gamma_0} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{\Gamma_0} \frac{f(\zeta)}{\zeta - z_0}\left(\frac{1}{1-\left(\frac{z-z_0}{\zeta - z_0}\right)}\right) + \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\zeta)}{z - z_0}\,\frac{1}{1-\frac{\zeta - z_0}{z - z_0}}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{\Gamma_0} \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{+\infty}\frac{(z-z_0)^n}{(\zeta-z_0)^n}\, d\zeta + \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\zeta)}{z - z_0}\sum_{n=0}^{+\infty}\frac{(\zeta-z_0)^n}{(z-z_0)^n}\, d\zeta$$

$$= \sum_{n=0}^{+\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_0}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta\right)(z-z_0)^n + \sum_{n=0}^{+\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_2} f(\zeta)(\zeta-z_0)^n\, d\zeta\right)\cdot (z-z_0)^{-n-1}$$

$$= \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

Here a_n is called Laurent series coefficients

$$a_n = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_0}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta & \text{if } n \geq 0 \\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_2}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta & \text{if } n < 0. \end{cases}$$

We claimed that the Laurent series expansion of $f(z)$ is unique. We also showed that Laurent series can be reduced to Taylor series if f is analytic in whole disk as an example. we considered

- Laurent series of $\frac{1}{z(1+z^2)}$ in $|z|<1$
- expansion of $e^{1/z}$.
- find all coefficient of $\frac{1}{(z-i)^2}$ in Laurent series expansion